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Quasi-periodic waves and an asymptotic property for the asymmetrical Nizhnik–Novikov–Veselov equation

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Abstract
Based on a multi-dimensional Riemann theta function, the Hirota bilinear method is extended to explicitly construct multi-periodic (quasi-periodic) wave solutions for the asymmetrical Nizhnik–Novikov–Veselov equation. Among these periodic waves, two-periodic waves are a direct generalization of well-known cnoidal waves; their surface pattern is two dimensional. The main physical result is the description of the behavior of nonlinear waves in shallow water. A limiting procedure is presented to analyze asymptotic properties of the two-periodic waves in details. Relations between the periodic wave solutions and the well-known soliton solutions are established. It is rigorously shown that the periodic wave solutions tend to the soliton solutions under a 'small amplitude' limit.

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(Some figures in this article are in colour only in the electronic version)

1. Introduction

It is well known that the bilinear derivative method developed by Hirota is a powerful and direct approach to constructing an exact solution of nonlinear equations. Once a nonlinear equation is written in bilinear forms by a dependent variable transformation, then multi-soliton solutions are usually obtained [1–7]. Nakamura proposed a convenient way to construct a kind of quasi-periodic solution of the nonlinear equation in his two serial papers [8, 9], where the periodic wave solutions of the KdV equation and the Boussinesq equation were obtained by means of Hirota's bilinear method. This method not only conveniently obtains periodic solutions of a nonlinear equation, but also directly gives explicit relations among frequencies, wavenumbers, phase shifts and amplitudes of the wave. Recently, we have extended this method to investigate the discrete Toda lattice [10]. In the present paper, we consider the following (2+1)-dimensional Nizhnik–Novikov–Veselov (ANNV) equation:

$$u_t + u_{xxx} + 3\left[u\int u_x\,\mathrm{d}y\right]_x = 0, \tag{1.1}$$

which was first derived by Boiti *et al* using a weak Lax pair context [11]. The ANNV equation can also be obtained from the inner parameter-dependent symmetry constraint of the KP equation and may be considered as a model for an incompressible fluid where u is a component of the velocity [12, 13]. In recent years, many papers have been focusing their topics on various exact solutions of equation (1.1) including soliton solutions, Jacobi or Weierstrass elliptic periodic solutions [16, 20]. However, these solutions are actually one-dimensional cnoidal waves. One of the major shortcomings of cnoidal theory as a practical model of water waves is that the theory is one dimensional, whereas the water surface is two dimensional. The quasi-periodic solutions of equation (1.1), which can be considered as a multi-dimensional generalization of cnoidal waves, are still unknown to our knowledge.

The objective of this paper is to construct two-periodic solutions of equation (1.1) and provide a detailed asymptotic analysis procedure for the solutions. This paper is organized as follows. In section 2, we briefly introduce a useful bilinear form of equation (1.1) and the Riemann theta function. In section 3, we apply Hirota's bilinear method to construct two-periodic wave solutions of equation (1.1). We further apply a limiting procedure to analyze the features and asymptotic behavior of the two-periodic wave solutions in details. It is rigorously shown that the periodic solutions tend to the known soliton solutions under a 'small amplitude' limit. Finally, we briefly discuss the conditions on the construction of multi-periodic wave solutions of equation (1.1) by using Hirota's bilinear method in section 4.

2. The bilinear form and the Riemann theta function

In this section, we briefly introduce a useful bilinear form of equation (1.1) and some main points on the Riemann theta function. By the dependent variable transformation [14, 15]

$$u = 2\partial_{xy}^2 \ln f(x, y, t),$$

equation (1.1) is then transformed into a bilinear form

$$\left(D_yD_t + D_yD_x^3\right) f(x, y, t)\cdot f(x, y, t) = 0, \tag{2.1}$$

where the bilinear differential operators D_x, D_y and D_t are defined by

$$D_x^m D_y^n D_t^k f(x, y, t)\cdot g(x, y, t) = (\partial_x - \partial_{x'})^m (\partial_y - \partial_{y'})^n \times(\partial_t - \partial_{t'})^k f(x, y, t) g(x', y', t')|_{x'=x, y'=y, t'=t}.$$

The bilinear operators have a good property when acting on exponential functions, namely

$$D_x^m D_y^n D_t^k\, \mathrm{e}^{\xi_1}\cdot \mathrm{e}^{\xi_2} = (\alpha_1-\alpha_2)^m(\rho_1-\rho_2)^n(\omega_1-\omega_2)^k\,\mathrm{e}^{\xi_1+\xi_2},$$

where $\xi_j = \alpha_j x + \rho_j y + \omega_j t + \delta_j$, $j = 1, 2$. More generally, we have

$$G(D_x, D_y, D_t)\,\mathrm{e}^{\xi_1}\cdot\mathrm{e}^{\xi_2} = G(\alpha_1-\alpha_2, \rho_1-\rho_2, \omega_1-\omega_2)\,\mathrm{e}^{\xi_1+\xi_2}, \tag{2.2}$$

where $G(D_x, D_y, D_t)$ is a polynomial about D_x, D_y and D_t. This property will be used later and plays a key role in the construction of the periodic wave solutions.

Following the Hirota bilinear theory, equation (1.1) admits a two-soliton solution:

$$u_2 = 2\partial_{xy}^2 \ln(1 + \mathrm{e}^{\eta_1} + \mathrm{e}^{\eta_2} + \mathrm{e}^{\eta_1+\eta_2+A_{12}}), \tag{2.3}$$

with

$$e^{A_{12}} = \frac{(\nu_1 - \nu_2)(\mu_1 - \mu_2)}{(\nu_1 + \nu_2)(\mu_1 + \mu_2)}, \qquad \eta_j = \mu_j x + \nu_j y - \mu_j^3 t + \gamma_j, \qquad j = 1, 2,$$

and here μ_j, ν_j, γ_j, $j = 1, 2$, are free constants.

To apply the Hirota bilinear method for constructing multi-periodic wave solutions of equation (1.1), we consider a slightly generalized form of the bilinear equation (2.1). Here we look for its solution in the form

$$u = u_0 + 2\partial_{xy}^2 \ln \vartheta(\xi), \tag{2.4}$$

where u_0 is a constant solution of equation (1.1) and the phase variable ξ is taken as the form $\xi = (\xi_1, \ldots, \xi_N)^T, \xi_j = \alpha_j x + \rho_j y + \omega_j t + \delta_j$, $j = 1, 2, \ldots, N$.

By substituting (2.4) into (1.1) and integrating with respect to x, we then get the following bilinear form:

$$G(D_x, D_y, D_t)\vartheta(\xi) \cdot \vartheta(\xi) = \left(D_y D_t + D_y D_x^3 + 3u_0 D_x^2 + c\right)\vartheta(\xi) \cdot \vartheta(\xi) = 0, \tag{2.5}$$

where $c = c(y, t)$ is an integration constant. For the bilinear equation (2.5), we are interested in its multi-periodic solutions in terms of the Riemann theta function $\vartheta(\xi)$.

Let us consider multi-periodic wave solutions of equation (1.1) based on the following multi-dimensional Riemann theta function of genus N:

$$\vartheta(\xi) = \vartheta(\xi, \tau) = \sum_{n \in Z^N} e^{-\pi\langle \tau n, n\rangle + 2\pi i\langle \xi, n\rangle}. \tag{2.6}$$

Here, the integer value vector $n = (n_1, \ldots, n_N)^T \in Z^N$ and complex phase variables $\xi = (\xi_1, \ldots, \xi_N)^T \in \mathcal{C}^N$. Moreover, for two vectors $f = (f_1, \ldots, f_N)^T$ and $g = (g_1, \ldots, g_N)^T$, their inner product is defined by

$$\langle f, g\rangle = f_1 g_1 + f_2 g_2 + \cdots + f_N g_N.$$

$\tau = (\tau_{ij})$ is a positive definite and real-valued symmetric $N \times N$ matrix, which we call the period matrix of the theta function. The entries τ_{ij} of the period matrix τ can be considered as free parameters of the theta function (2.6).

In the simplest case when $N = 1$, solution (2.5) reproduces the cnoidal waves, which is actually the Weierstrass or Jacobi elliptic solution (for example, see [16–20]) according to the following relations:

$$\begin{aligned}
&\wp(\xi, \tau) = -(\ln \vartheta_{11}(\xi, \tau))'' + c,\\
&cn(\pi\vartheta(0, \tau)\xi, k) = \frac{\vartheta_{01}(0, \tau)\vartheta_{10}(\xi, \tau)}{\vartheta_{10}(0, \tau)\vartheta_{01}(\xi, \tau)}, \qquad k = \left(\frac{\vartheta_{10}(0, \tau)}{\vartheta(0, \tau)}\right)^2,
\end{aligned}$$

where c is defined so that the Laurent expansion of $\wp(\xi, \tau)$ at $\xi = 0$ has a zero constant term and k is called the modulus of the Jacobi elliptic function. Three auxiliary (or half-period) theta functions are defined by

$$\begin{aligned}
&\vartheta_{01}(\xi, \tau) = \vartheta\left(\xi + \tfrac{1}{2}, \tau\right),\\
&\vartheta_{10}(\xi, \tau) = e^{-\frac{1}{4}\pi\tau + i\pi\xi}\vartheta\left(\xi + i\tfrac{1}{2}\tau, \tau\right),\\
&\vartheta_{11}(\xi, \tau) = e^{-\frac{1}{4}\pi\tau + i\pi(\xi + \frac{1}{2})}\vartheta\left(\xi + i\tfrac{1}{2}\tau + \tfrac{1}{2}, \tau\right).
\end{aligned}$$

So the waves of interest in this paper appear at the case when $N = 2$; solution (2.4) is then periodic in two independent horizontal directions.

3. Two-periodic waves and asymptotic properties

In this section, we consider two-periodic wave solutions to equation (1.1), which are a two-dimensional generalization of one-periodic wave solutions. The two-periodic waves of interest here have three-dimensional velocity fields and two-dimensional surface patterns.

3.1. Construction of two-periodic waves

In the case when $N = 2$, the Riemann theta function (2.6) takes the form

$$\vartheta(\xi, \tau) = \vartheta(\xi_1, \xi_2, \tau) = \sum_{n\in Z^2} \mathrm{e}^{2\pi\mathrm{i}\langle\xi,n\rangle-\pi\langle\tau n,n\rangle}, \tag{3.1}$$

where $n = (n_1, n_2)^T \in Z^2, \xi = (\xi_1, \xi_2)^T \in \mathcal{C}^2, \xi_i = \alpha_j x + \rho_j y + \omega_j t + \delta_j,\ j = 1, 2$. τ is a positive definite and real-valued symmetric 2×2 matrix which can take the form

$$\tau = \begin{pmatrix} \tau_{11} & \tau_{12} \\ \tau_{12} & \tau_{22} \end{pmatrix}, \qquad \tau_{11} > 0, \qquad \tau_{22} > 0, \qquad \tau_{11}\tau_{22} - \tau_{12}^2 > 0.$$

In order to get some sufficient conditions, such that the theta function (3.1) satisfies the bilinear equation (2.5), we substitute function (3.1) into the left of equation (2.5) and obtain that

$$\begin{aligned}
&G(D_x, D_y, D_t)\vartheta(\xi_1, \xi_2, \tau)\cdot\vartheta(\xi_1, \xi_2, \tau)\\
&\quad= \sum_{m,n\in Z^2} G(2\pi\mathrm{i}\langle n-m,\alpha\rangle, 2\pi\mathrm{i}\langle n-m,\rho\rangle, 2\pi\mathrm{i}\langle n-m,\omega\rangle)\,\mathrm{e}^{2\pi\mathrm{i}\langle\xi,n+m\rangle-\pi(\langle\tau m,m\rangle+\langle\tau n,n\rangle)}\\
&\quad\overset{m=m'-n}{=} \sum_{m'\in Z^2}\sum_{n\in Z^2} G(2\pi\mathrm{i}\langle 2n-m',\alpha\rangle, 2\pi\mathrm{i}\langle 2n-m',\rho\rangle, 2\pi\mathrm{i}\langle 2n-m',\omega\rangle)\\
&\qquad\times\exp\{-\pi(\langle\tau(n-m'), n-m'\rangle + \langle\tau n,n\rangle)\}\exp\{2\pi\mathrm{i}\langle\xi,m'\rangle\}\\
&\quad\equiv \sum_{m'\in Z^2}\bar{G}(m_1', m_2')\,\mathrm{e}^{2\pi\mathrm{i}\langle\xi,m'\rangle}.
\end{aligned}$$

In the last line, we have introduced the notation $\bar{G}(m_1', m_2')$ for the coefficient of $\mathrm{e}^{2\pi\mathrm{i}\langle\xi,m'\rangle}$. For each fixed $l = 1, 2$, by shifting the jth summation index as $n_j = n_j' + \delta_{j,l}$ with $\delta_{j,l}$ representing Kronecker's delta, we obtain that

$$\begin{aligned}
&\bar{G}(m_1', m_2')\\
&\quad= \sum_{n\in Z^2} G(2\pi\mathrm{i}\langle 2n-m',\alpha\rangle, 2\pi\mathrm{i}\langle 2n-m',\rho\rangle, 2\pi\mathrm{i}\langle 2n-m',\omega\rangle)\,\mathrm{e}^{-\pi(\langle\tau(n-m'),n-m'\rangle+\langle\tau n,n\rangle)}\\
&\quad= \sum_{n\in Z^2} G\left(2\pi\mathrm{i}\sum_{j=1}^{2}(2n_j' - (m_j' - 2\delta_{jl}))\alpha_j, 2\pi\mathrm{i}\sum_{j=1}^{2}(2n_j' - (m_j' - 2\delta_{jl}))\rho_j,\right.\\
&\qquad\left. 2\pi\mathrm{i}\sum_{j=1}^{2}(2n_j' - (m_j' - 2\delta_{jl}))\omega_j\right)\exp\left\{-\pi\sum_{j,k=1}^{2}(n_j' + \delta_{jl})\tau_{jk}(n_k' + \delta_{kl})\right.\\
&\qquad\left. -\pi\sum_{j,k=1}^{2}[(m_j' - 2\delta_{jl} - n_j') + \delta_{jl}]\tau_{jk}[(m_k' - 2\delta_{kl} - n_k') + \delta_{kl}]\right\},\\
&\quad= \begin{cases} \bar{G}(m_1' - 2, m_2')\,\mathrm{e}^{-2\pi(\tau_{11}m_1'+\tau_{12}m_2')+2\pi\tau_{11}}, & l = 1,\\ \bar{G}(m_1', m_2' - 2)\,\mathrm{e}^{-2\pi(\tau_{12}m_1'+\tau_{22}m_2')+2\pi\tau_{22}}, & l = 2, \end{cases}
\end{aligned}$$

which implies that if the following equations are satisfied:

$$\bar{G}(0,0) = \bar{G}(0,1) = \bar{G}(1,0) = \bar{G}(1,1) = 0, \tag{3.2}$$

then we have $\bar{G}(m_1', m_2') = 0$ for all $m_1', m_2' \in Z$, and thus function (3.1) is an exact solution of equation (2.5).

By introducing the notations as

$$\begin{aligned}
&M = (a_{jl}), \qquad b = (b_1, b_2, b_3, b_4)^T,\\
&a_{j1} = -4\pi^2 \sum_{n_1,n_2\in Z^2} \langle 2n - s^j, \rho\rangle \big(2n_1 - s_1^j\big)\varepsilon_j(n),\\
&a_{j2} = -4\pi^2 \sum_{n_1,n_2\in Z^2} \langle 2n - s^j, \rho\rangle \big(2n_2 - s_2^j\big)\varepsilon_j(n)\\
&a_{j3} = -12\pi^2 \sum_{n_1,n_2\in Z^2} \langle 2n - s^j, \alpha\rangle \langle 2n - s^j, \alpha\rangle\varepsilon_j(n),\\
&a_{j4} = \sum_{n_1,n_2\in Z^2} \varepsilon_j(n),\\
&b_j = -16\pi^4 \sum_{n_1,n_2\in Z^2} \langle 2n - s^j, \alpha\rangle^3 \langle 2n - s^j, \rho\rangle\varepsilon_j(n),\\
&\varepsilon_j(n) = \lambda_1^{n_1^2+(n_1-s_1^j)^2}\lambda_2^{n_2^2+(n_2-s_2^j)^2}\lambda_3^{n_1n_2+(n_1-s_1^j)(n_2-s_2^j)},\\
&\lambda_1 = \mathrm{e}^{-\pi\tau_{11}}, \qquad \lambda_2 = \mathrm{e}^{-\pi\tau_{22}}, \qquad \lambda_3 = \mathrm{e}^{-2\pi\tau_{12}},\\
&s^j = \big(s_1^j, s_2^j\big), \qquad j = 1, 2, 3, 4,\\
&s^1 = (0,0), \qquad s^2 = (1,0), \qquad s^3 = (0,1), \qquad s^4 = (1,1),
\end{aligned}$$

equation (3.2) can be written as a linear system:

$$M(\omega_1, \omega_2, u_0, c)^T = b. \tag{3.3}$$

Hence, we get an exact two-periodic wave solution to equation (1.1):

$$u = u_0 + 2\partial_{xy}^2 \ln\vartheta(\xi_1, \xi_2, \tau), \tag{3.4}$$

with $\vartheta(\xi_1, \xi_2)$ and $\omega_1, \omega_2, u_0, c$ being given by (3.1) and (3.3) respectively, while other parameters $\alpha_1, \alpha_2, \rho_1, \rho_2, \tau_{11}, \tau_{22}, \tau_{12}$ are free. The two-periodic wave is specified by six of the parameters $\alpha_1, \alpha_2, \rho_1, \rho_2, \tau_{11}$ and τ_{22}.

3.2. *Feature and asymptotic property of two-periodic waves*

The two-periodic wave (3.4) has a simple characterization as follows.

(i) It is a direct generalization of one-periodic waves; its surface pattern is two dimensional, i.e. there are two phase variables ξ_1 and ξ_2 respectively. It has two independent spatial periods in two independent horizontal directions. The two-periodic wave may be considered to represent periodic waves in shallow water without the assumption of one dimensionality.

(ii) It has $2N$ fundamental periods $\{e_j, j = 1, \ldots, N\}$ and $\{\mathrm{i}\tau_j, j = 1, \ldots, N\}$ in (ξ_1, ξ_2). Its velocity of propagation is given by

$$\frac{\mathrm{d}x}{\mathrm{d}t} = \frac{\omega_2\alpha_1 - \omega_1\alpha_2}{\alpha_1\rho_2 - \alpha_2\rho_1}, \qquad \frac{\mathrm{d}y}{\mathrm{d}t} = \frac{\omega_1\rho_2 - \omega_2\rho_1}{\alpha_1\rho_2 - \alpha_2\rho_1}.$$

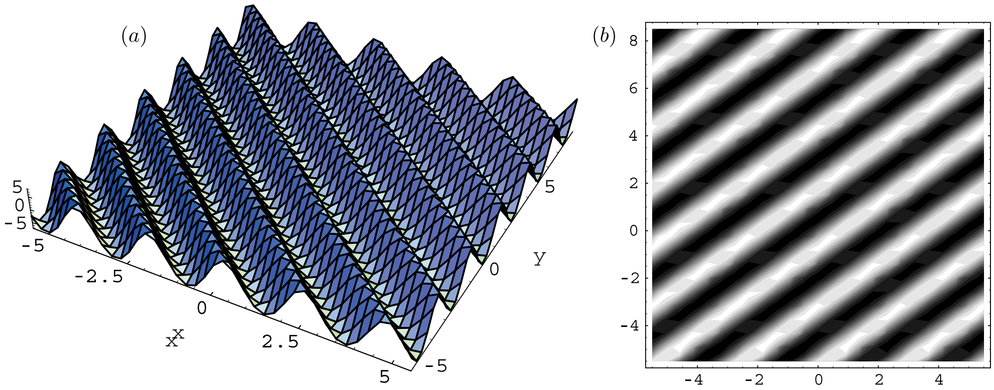

Figure 1. A degenerate two-periodic wave to the ANNV equation with parameters $\frac{\alpha_2}{\alpha_1} = \frac{\rho_2}{\rho_1}$ and $\alpha_1 = 0.1, \alpha_2 = 3, \tau_{11} = 0.2, \tau_{12} = -0.3, \tau_{22} = 1, \rho_1 = 0.01, \rho_2 = 0.3$. This figure shows that the degenerate two-periodic wave is almost one dimensional. (*a*) Perspective view of the wave and (*b*) overhead view of the wave, with the contour plot being shown. The bright lines are crests and the dark lines are troughs.

(iii) If parameters satisfy a ratio relation

$$\frac{\alpha_2}{\alpha_1} = \frac{\rho_2}{\rho_1} = k \quad (k \text{ is a constant}),$$

then

$$\omega_2 \sim k\omega_1, \qquad \xi_2 \sim k\xi_1, \qquad \vartheta(\xi_1, \xi_2) \sim \vartheta(\xi_1, k\xi_1).$$

Therefore, the two-periodic wave is actually one dimensional and it degenerates to a one-periodic wave (see figure 1).

(iv) If parameters do not satisfy a ratio relation, that is,

$$\frac{\alpha_2}{\alpha_1} \neq \frac{\rho_2}{\rho_1},$$

then for any time t, phase variables $\xi_1 = \text{const}$ and $\xi_2 = \text{const}$ intersect at a unique point. As the time t changes, this point moves in the (x, y) plane with a constant speed. In this case, the two-periodic solution is genuinely two dimensional, and it is spatially periodic in two independent directions in the (x, y) plane. Every two-periodic wave like figure 2 is spatially periodic in two directions, but it need not be periodic in either the x- or y-direction. The basic cell of the pattern seems like a hexagon, but need not be regular: six steep wave crests form the edges of each hexagon. The six crests surrounding a trough can be identified in pairs: opposite crests are parallel and have equal amplitudes as well as lengths along the crests.

(v) In a subcase of the above: $\tau_{11} = \tau_{22}, \alpha_1 = \alpha_2, \rho_1 = -\rho_2$, the two-periodic solution has only three independent parameters $(\tau_{11}, \alpha_1, \rho_1)$, and it is called a symmetric solution. This solution is periodic in both x- and y-directions and propagate purely in the x-direction. An example is shown in figure 3. It is seen that the cell of its pattern is a regular hexagon from the contour plot (see figure 3(*b*)).

Finally, we consider the asymptotic properties of the two-periodic solution (3.4). In a way similar to theorem 1, we can establish the relation between the two-periodic solution (3.4) and the two-soliton solution (2.3) as follows.

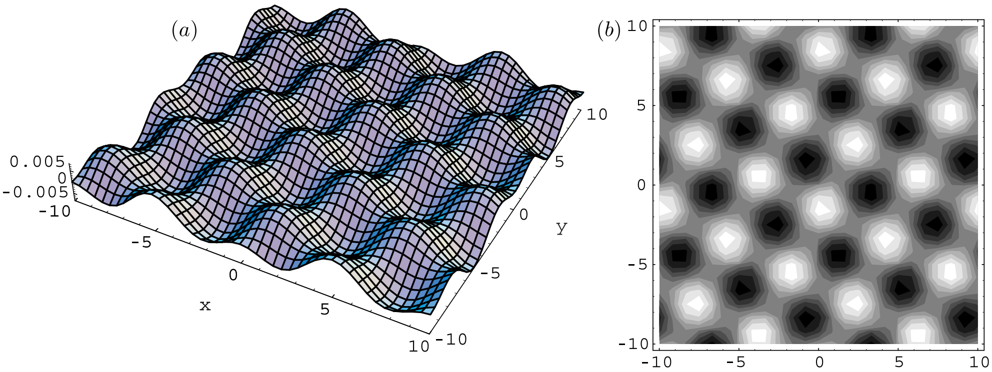

Figure 2. An asymmetric two-periodic wave for the ANNV equation with parameters: $\alpha_1 = 0.1, \alpha_2 = 0.2, \tau_{11} = 2, \tau_{12} = 0.2, \tau_{22} = 2, \rho_1 = 0.2, \rho_2 = -0.1$. This figure shows that every two-periodic wave is spatially periodic in two directions, but it need not be periodic in either the x- or y-direction. (a) Perspective view of the wave and (b) overhead view of the wave, with the contour plot being shown. The bright hexagons are crests and the dark hexagons are troughs.

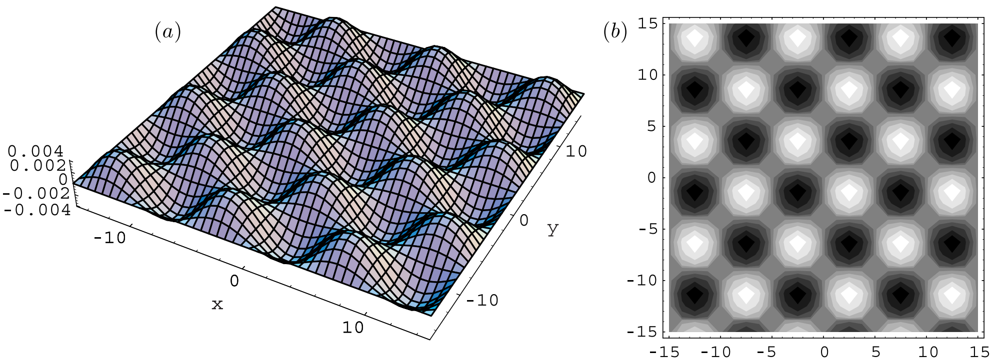

Figure 3. A symmetric two-periodic wave for the ANNV equation with parameters: $\alpha_1 = 0.1$, $\alpha_2 = 0.1, \tau_{11} = 2, \tau_{12} = 0.2, \tau_{22} = 1, \rho_1 = 0.1, \rho_2 = -0.1$. This figure show that the symmetric two-periodic wave is periodic both in x- and y-directions and propagate purely in the x-direction. (a) Perspective view of the wave and (b) overhead view of the wave, with the contour plot being shown. The bright hexagons are crests and the dark hexagons are troughs.

Theorem 1. *Assume that $(\omega_1, \omega_2, u_0, c)^T$ is a solution of system (3.3), and for the two-periodic wave solution (3.4), we take*

$$\begin{aligned} &\alpha_j = \frac{\mu_j}{2\pi \mathrm{i}}, \qquad \rho_j = \frac{\nu_j}{2\pi \mathrm{i}}, \qquad \delta_j = \frac{\gamma_j + \pi\tau_{jj}}{2\pi \mathrm{i}}, \\ &\tau_{12} = \frac{A_{12}}{2\pi \mathrm{i}}, \qquad j = 1, 2, \end{aligned} \tag{3.5}$$

with μ_j, ν_j, γ_j, $j = 1, 2$, and A_{12} being as those given in (2.3). Then we have the following asymptotic relations:

$$\begin{aligned} &u_0 \longrightarrow 0, \qquad c \longrightarrow 0, \qquad \xi_j \longrightarrow \frac{\eta_j + \pi\tau_{jj}}{2\pi \mathrm{i}}, \qquad j = 1, 2, \\ &\vartheta(\xi_1, \xi_2, \tau) \longrightarrow 1 + \mathrm{e}^{\eta_1} + \mathrm{e}^{\eta_2} + \mathrm{e}^{\eta_1+\eta_2+A_{12}}, \\ &\text{as} \qquad \lambda_1, \lambda_2 \to 0. \end{aligned} \tag{3.6}$$

So the two-periodic solution (3.4) just tends to the two-soliton solution (2.3) under a certain limit, namely

$$u \longrightarrow u_2, \qquad \text{as} \quad \lambda_1, \lambda_2 \to 0.$$

Proof. We expand the periodic wave function $\vartheta(\xi_1, \xi_2)$ in the following form:

$$\vartheta(\xi_1,\xi_2,\tau) = 1 + (\mathrm{e}^{2\pi \mathrm{i}\xi_1} + \mathrm{e}^{-2\pi\mathrm{i}\xi_1})\,\mathrm{e}^{-\pi\tau_{11}} + (\mathrm{e}^{2\pi\mathrm{i}\xi_2} + \mathrm{e}^{-2\pi\mathrm{i}\xi_2})\,\mathrm{e}^{-\pi\tau_{22}} + (\mathrm{e}^{2\pi\mathrm{i}(\xi_1+\xi_2)} + \mathrm{e}^{-2\pi\mathrm{i}(\xi_1+\xi_2)})\,\mathrm{e}^{-\pi(\tau_{11}+2\tau_{12}+\tau_{22})} + \cdots.$$

Further by using (3.5) and making a transformation $\hat{\omega}_j = 2\pi\mathrm{i}\omega_j$, $j = 1, 2$, we get

$$\begin{aligned}\vartheta(\xi_1,\xi_2,\tau) &= 1 + \mathrm{e}^{\hat{\xi}_1} + \mathrm{e}^{\hat{\xi}_2} + \mathrm{e}^{\hat{\xi}_1+\hat{\xi}_2-2\pi\tau_{12}} + \lambda_1^2\,\mathrm{e}^{-\hat{\xi}_1} \\ &\quad + \lambda_2^2\,\mathrm{e}^{-\hat{\xi}_2} + \lambda_1^2\lambda_2^2\,\mathrm{e}^{-\hat{\xi}_1-\hat{\xi}_2-2\pi\tau_{12}} + \cdots \\ &\longrightarrow 1 + \mathrm{e}^{\hat{\xi}_1} + \mathrm{e}^{\hat{\xi}_2} + \mathrm{e}^{\hat{\xi}_1+\hat{\xi}_2+A_{12}}, \qquad \text{as} \quad \lambda_1,\lambda_2 \to 0,\end{aligned}$$

where $\hat{\xi}_j = \mu_j x + \nu_j y + \hat{\omega}_j t + \gamma_j$, $j = 1, 2$.

It remains to prove that

$$c \longrightarrow 0, \qquad \hat{\omega}_j \longrightarrow -\mu_j^3, \qquad \hat{\xi}_j \longrightarrow \eta_j, \qquad j = 1, 2, \qquad \text{as} \quad \lambda_1, \lambda_2 \to 0. \tag{3.7}$$

As in (3.15), we can expand each function in $\{a_{ij}, b_j, j = 1, 2, 3, 4\}$ into a series with λ_1, λ_2. It is slightly more tedious than (3.15), but this process is easily carried out by using symbolic computation software Mathematica or Maple. Actually, we only need to make the first-order expansions of matrix M and vector b with λ_1, λ_2 to show the asymptotic relations (3.7). Here we consider their second-order expansions to see deeper relations among parameters for the two-periodic solution (3.4) and the two-soliton solution (2.3). The expansions for the matrix M and the vector b are given by

$$\begin{aligned} M &= \begin{pmatrix} 0&0&0&1\\0&0&0&0\\0&0&0&0\\0&0&0&0\end{pmatrix} + \begin{pmatrix} 0&0&0&0\\-8\pi^2\rho_1&0&-24\pi^2\alpha_1^2&2\\0&0&0&0\\0&0&0&0\end{pmatrix}\lambda_1 \\ &\quad + \begin{pmatrix} 0&0&0&0\\0&0&0&0\\0&-8\pi^2\rho_2&-24\pi^2\alpha_2^2&2\\0&0&0&0\end{pmatrix}\lambda_2 + \begin{pmatrix} -32\pi^2\rho_1&0&-96\pi^2\alpha_1^2&2\\0&0&0&0\\0&0&0&0\\0&0&0&0\end{pmatrix}\lambda_1^2 \\ &\quad + \begin{pmatrix} 0&-32\pi^2\rho_2&-96\pi^2\alpha_2^2&2\\0&0&0&0\\0&0&0&0\\0&0&0&0\end{pmatrix}\lambda_2^2 \\ &\quad + \begin{pmatrix} 0&0&0&0\\0&0&0&0\\0&0&0&0\\-8\pi^2(\rho_1-\rho_2)&8\pi^2(\rho_1-\rho_2)-8\pi^2(\rho_1+\rho_2)\lambda_3&-24\pi^2(\alpha_2-\alpha_1)^2&2\end{pmatrix}\lambda_1\lambda_2 \\ &\quad + o\big(\lambda_1^k\lambda_2^j\big), \qquad k + j \geqslant 2, \end{aligned} \tag{3.8}$$

and

$$b = \begin{pmatrix}0\\-32\pi^4\alpha_1^3\rho_1\\0\\0\end{pmatrix}\lambda_1 + \begin{pmatrix}0\\0\\-32\pi^4\alpha_2^3\rho_2\\0\end{pmatrix}\lambda_2 + \begin{pmatrix}-512\pi^4\alpha_1^3\rho_1\\0\\0\\0\end{pmatrix}\lambda_1^2 + \begin{pmatrix}0\\-512\pi^4\alpha_2^3\rho_2\\0\\0\\0\end{pmatrix}\lambda_2^2$$

$$+\begin{pmatrix} 0 \\ 0 \\ 0 \\ -32\pi^4(\alpha_1+\alpha_2)^3(\rho_1+\rho_2)\lambda_3 - 32\pi^4(\alpha_1-\alpha_2)^3(\rho_1-\rho_2) \end{pmatrix}\lambda_1\lambda_2$$
$$+ o\left(\lambda_1^k\lambda_2^j\right), \qquad k+j \geqslant 2, \tag{3.9}$$

where $o\left(\lambda_1^k\lambda_2^j\right)$ denote higher infinitesimal than $\lambda_1^k\lambda_2^j$, $k+j \geqslant 2$.

We also assume the solution of system (3.3) in the following form:

$$\begin{pmatrix} \omega_1 \\ \omega_2 \\ u_0 \\ c \end{pmatrix} = \begin{pmatrix} \omega_1^{(0)} \\ \omega_2^{(0)} \\ u_0^{(0)} \\ c^{(0)} \end{pmatrix} + \begin{pmatrix} \omega_1^{(1)} \\ \omega_2^{(1)} \\ u_0^{(1)} \\ c^{(1)} \end{pmatrix}\lambda_1 + \begin{pmatrix} \omega_1^{(2)} \\ \omega_2^{(2)} \\ u_0^{(2)} \\ c^{(2)} \end{pmatrix}\lambda_2 + \begin{pmatrix} \omega_1^{(11)} \\ \omega_2^{(11)} \\ u_0^{(11)} \\ c^{(11)} \end{pmatrix}\lambda_1^2$$
$$+ \begin{pmatrix} \omega_1^{(22)} \\ \omega_2^{(22)} \\ u_0^{(22)} \\ c^{(22)} \end{pmatrix}\lambda_2^2 + \begin{pmatrix} \omega_1^{(12)} \\ \omega_2^{(12)} \\ u_0^{(12)} \\ c^{(12)} \end{pmatrix}\lambda_1\lambda_2 + o\left(\lambda_1^k\lambda_2^j\right), \qquad k+j \geqslant 2. \tag{3.10}$$

Substituting (3.8)–(3.10) into (3.3) and comparing the same order of λ_1, λ_2, we obtain the following relations:

$$\begin{aligned}
&c^{(0)} = c^{(1)} = c^{(2)} = c^{(12)} = 0, \\
&\rho_1\omega_1^{(0)} + 3\alpha_1^2 u_0^{(0)} = 4\pi^2\alpha_1^3\rho_1, \qquad \rho_2\omega_2^{(0)} + 3\alpha_2^2 u_0^{(0)} = 4\pi^2\alpha_2^3\rho_2, \\
&\rho_1\omega_1^{(1)} + 3\alpha_1^2 u_0^{(1)} = 0, \qquad \rho_2\omega_2^{(1)} + 3\alpha_2^2 u_0^{(1)} = 0, \\
&c^{(11)} - 32\pi^2\rho_1\omega_1^{(0)} - 96\pi^2\alpha_1^2 u_0^{(0)} = -512\pi^4\alpha_1^3\rho_1, \\
&c^{(22)} - 32\pi^2\rho_2\omega_2^{(0)} - 96\pi^2\alpha_2^2 u_0^{(0)} = -512\pi^4\alpha_2^3\rho_2, \\
&\ldots .
\end{aligned}$$

To make relations (3.7) hold, we choose $u_0^{(0)} = \omega_1^{(1)} = \omega_2^{(1)} = 0$, and thus

$$\begin{aligned}
&u_0 = o(\lambda_1, \lambda_2) \longrightarrow 0, \\
&c = -384\pi^4\alpha_1^3\rho_1\lambda_1^2 - 384\pi^4\alpha_2^3\rho_2\lambda_2^2 + o\left(\lambda_1^2, \lambda_2^2\right) \longrightarrow 0, \\
&\omega_1 = 4\pi^2\alpha_1^3 + o(\lambda_1, \lambda_2) \longrightarrow 4\pi^2\alpha_1^3, \\
&\omega_2 = 4\pi^2\alpha_2^3 + o(\lambda_1, \lambda_2) \longrightarrow 4\pi^2\alpha_2^3, \qquad \text{as} \quad \lambda_1, \lambda_2 \to 0,
\end{aligned}$$

which implies (3.7). Therefore, we conclude that the two-periodic solution (3.4) tends to the two-soliton solution (2.3) as $\lambda_1, \lambda_2 \to 0$. □

4. Discussion on the conditions of *N*-periodic wave solutions

In this section, we consider a condition for an N-periodic wave solution of equation (1.1). The theta function takes the form

$$\vartheta(\xi, \tau) = \vartheta(\xi_1, \ldots, \xi_N, \tau) = \sum_{n \in Z^N} e^{2\pi i\langle \xi, n\rangle - \pi\langle \tau n, n\rangle}, \tag{4.1}$$

where $n = (n_1, \ldots, n_N)^T \in Z^N$, $\xi = (\xi_1, \ldots, \xi_N)^T \in \mathcal{C}^N$, $\xi_i = \alpha_j x + \rho_j y + \omega_j t + \delta_j$, $j = 1, \ldots, N$ and τ is a $N \times N$ symmetric positive definite matrix.

In order to get the conditions, such that function (4.1) satisfies the bilinear equation (2.5), we substitute (4.1) into the left of equation (2.5) and obtain

$$
\begin{aligned}
G(D_x, D_y, D_t)\vartheta(\xi_1, \ldots, \xi_N, \tau) \cdot \vartheta(\xi_1, \ldots, \xi_N, \tau)
&= \sum_{m,n\in Z^N} G(2\pi \mathrm{i}\langle n-m, \alpha\rangle, 2\pi \mathrm{i}\langle n-m, \rho\rangle, 2\pi \mathrm{i}\langle n-m, \omega\rangle) \\
&\quad \times \exp(2\pi \mathrm{i}\langle \xi, n+m\rangle - \pi(\langle \tau m, m\rangle + \langle \tau n, n\rangle)) \\
&\equiv \sum_{m'\in Z^N} \bar{G}(m_1', \ldots, m_N')\, \mathrm{e}^{2\pi \mathrm{i}\langle \xi, m'\rangle}.
\end{aligned}
$$

In the last line, we have introduced the notation $\bar{G}(m_1', \ldots, m_N')$ for the coefficient of $\mathrm{e}^{2\pi \mathrm{i}\langle \xi, m'\rangle}$. By shifting the j th summation index as $n_j = n_j' + \delta_{j,l}$ with $\delta_{j,l}$ representing Kronecker's delta, we obtain that

$$
\begin{aligned}
\bar{G}(m_1', \ldots, m_N') &= \sum_{n\in Z^N} G(2\pi \mathrm{i}\langle 2n-m', \alpha\rangle, 2\pi \mathrm{i}\langle 2n-m', \rho\rangle, 2\pi \mathrm{i}\langle 2n-m', \omega\rangle) \\
&\quad \times \exp\{-\pi(\langle \tau(n-m'), n-m'\rangle) + \langle \tau n, n\rangle\} \\
&= \bar{G}(m_1', \ldots, m_j - 2, \ldots, m_N') \times \exp\left(-2\pi \sum_{k=1}^{N} \tau_{kj} m_j' + 2\pi \tau_{jj}\right).
\end{aligned}
$$

This implies that if

$$
\bar{G}(m_1, \ldots, m_N') = 0, \tag{4.2}
$$

for all possible combinations of $m_1, \ldots, m_N = 0, 1$, then we have $\bar{G}(m_1', \ldots, m_N') = 0$ for all $m_1', \ldots, m_N' \in Z$, and thus function (4.1) is an exact solution of the bilinear equation (2.5). For the asymptotic property, the N-periodic wave solutions of the KdV equation going to its corresponding N-soliton solutions ever was described in Mumford's book [21].

Now we consider the number of equations and some unknown parameters. Obviously, the number of equations of the type (4.2) is 2^N. On the other hand, we have parameters $\tau_{ij} = \tau_{ji}, c, u_0, \alpha_i, \rho_i, \omega_i$, whose total number is $\frac{1}{2}N(N+1) + 2N + 2$. Among them, $2N$ parameters $\tau_{ii}, \alpha_1, \rho_i$ are taken to be the given parameters related to the amplitudes and wavenumbers (or frequencies) of N-periodic waves. Hence, the number of the unknown parameters is $\frac{1}{2}N(N+1)+2$. The number of equations is greater than the unknown parameters in the case when $N \leqslant 4$. This fact means that if equation (4.2) is satisfied by the unknowns, we have at least N-periodic wave solutions ($N \leqslant 4$). In this paper, we consider a two-periodic wave solution to equation (1.1), which belongs to the cases when $N = 2$. There are still certain numerical difficulties in the calculation for the case $N > 2$, which will be considered in our future work.

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